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# On the Mathematical Basis of the Dirac Formulation of Quantum Mechanics

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We discuss various attempts to implement mathematically the Dirac formulation of Quantum Mechanics. A first attempt used Hilbert space. This formalization realizes the Dirac formalism if and only if the spectra of the observables under consideration is purely discrete. Therefore, generalized spectral decompositions are needed. These spectral decompositions can be constructed in the framework of rigged Hilbert spaces. We construct generalized spectral decompositions for self-adjoint operators using their spectral measures. We review the previous work by Marlow (in Hilbert spaces), Antoine, Roberts, and Melsheimer and complete it. We show that these generalized spectral decompositions fit well in the framework of a theory constructed by Kato and Kuroda and that all the results can be reproduced in this framework.

**KEY WORDS:** spectral measures; eigen-function expansions; rigged Hilbert spaces.

## 1. INTRODUCTION

This paper is a revision of the mathematical foundations of the Dirac formulation of Quantum Mechanics. On this subject, there has been written quite a few contributions from several points of view: direct integrals of Hilbert spaces (Marlow, 1965), rigged Hilbert spaces (Antoine, 1969, 1998; Bohm, 1967, 1994; Melsheimer, 1974; Roberts, 1966a,b) and trajectory spaces (Eijndhoven and de Graaf, 1986). This paper is a unified version of Hilbert space and rigged Hilbert space mathematical implementation of the Dirac formulation. This implementation constructs suitable rigged rigged Hilbert spaces using spectral measures and direct integrals of Hilbert spaces as main tools.

Mathematical Physicists are used to working on the von Neumann mathematical formalism for Quantum Mechanics based in Hilbert space (von Neumann, 1955). However, it is well known that von Neumann mathematics do not fulfill a

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crucial requirement of Dirac: that any observable $A$ had a complete set of eigenvectors, with real eigenvalues on the spectrum of $A$ (the possible values of a measurement of $A$) so that any pure state be a linear combination (finite or infinite) of these eigenvectors. Von Neumann approach is only good if the system under study has a *purely discrete spectrum,* as it happens with atomic systems. However, for systems allowing observables with continous spectrum, like those involved in scattering processes, von Neumann theory does not justify the Dirac formulation.

This is the main motivation for alternative mathematizations of the Dirac formalism other than Hilbert spaces. There are, nevertheless, other arguments. For instance, rigged Hilbert space formulations of Quantum Mechanics, allow for a rigorous definition of vector states for resonances (Bohm, 1994; Bohm and Gadella, 1989) and for a conceptual development of irreversibility in Quantum Mechanics (Antoniou and Prigogine, 1993; Antoniou, Tasaki, 1992, 1993) among other things.

The traditional theory of rigged Hilbert spaces uses a triplet $\Phi \subset \mathcal{H} \subset \Phi^{\times}$ for which $\mathcal{H}$ is a Hilbert space, $\Phi$ is a locally convex topological vector space *dense in* $\mathcal{H}$ and $\Phi^{\times}$, the antidual of $\Phi$. On the other hand, the Kato–Kuroda (Kato and Kuroda, 1993) formalism shows that using a particular version of the spectral representations, the denseness of $\Phi$ is no longer necessary. In any case, we give a procedure to obtain eigenfunction (with respect to any observable $A$) expansions for vectors in the test space $\Phi$, that we shall always assume dense in $\mathcal{H}$. In the formalism proposed by Kato and Kuroda, it is sufficient that $\Phi$ be a generator with respect to the spectral measure. Although these ideas are not all new, we present them into a unified context and we present few new results like the explicit form of the measures and the generalized eigenvectors in terms of the spectral measures.

This paper is organized as follows: In Section 2, we make a brief review of the concept of observable according to Dirac. In Section 3, we review the notion of spectral measures and direct integrals of Hilbert spaces. In Section 4, we review the main results of the Dirac formulation in rigged Hilbert spaces. In Section 5, we discuss the notion of spectral forms and spectral representations and their properties. These constructions allow for new equipments of a spectral measure, without the hypothesis of nuclearity. These new versions of the nuclear spectral theorem present universal equipments, the minimal equipments represent the formalism in the Hilbert space (as Hilbert spaces are special forms of equipments) and the Kato-Kuroda formalism is included as a special form of equipment. This is presented in Section 6.

Theorems and Propositions are all together numbered in correlative order.

## 2. DIRAC KETS

To begin with, let us summarize the main features of the Dirac formulation of Quantum Mechanics. These are (Dirac, 1958):

– Each observable $A$ has a complete system of eigenkets $\{|\lambda\rangle\}$ whose respective eigenvalues cover the spectrum $\sigma(A)$ of $A$, which is the set of all possible results of a measure of the observable $A$, i.e.,

$$A|\lambda\rangle = \lambda|\lambda\rangle.$$

– The completeness of the system $\{|\lambda\rangle\}$ means that there exists a measure $\mu$ on $\sigma(A)$ such that for each ket $|\phi\rangle$ and each bra $\langle\varphi|$ we have the following Parseval type identity:

$$\langle\varphi|\phi\rangle = \int_{\sigma(A)} \langle\varphi|\lambda\rangle\langle\lambda|\phi\rangle \, d\mu(\lambda)$$

(We write $\langle\varphi|\lambda\rangle := \langle\varphi|\lambda\rangle^*$ where the star denotes complex conjugation.) If we omit the arbitrary bra $\langle\varphi|$ we have

$$|\phi\rangle = \int_{\sigma(A)} |\lambda\rangle\langle\lambda|\phi\rangle \, d\mu(\lambda) \tag{1}$$

– The observable $A$ admits the following integral form

$$A = \int_{\sigma(A)} \lambda|\lambda\rangle\langle\lambda| \, d\mu(\lambda)$$

that should be interpreted in the sense that for suitable kets $|\phi\rangle$ and bras $\langle\varphi|$, one has

$$\langle\varphi|A|\phi\rangle = \int_{\sigma(A)} \lambda\langle\varphi|\lambda\rangle\langle\lambda|\phi\rangle \, d\mu(\lambda) \tag{2}$$

– For each measurable Borel function $f(\lambda) : \sigma(A) \mapsto \mathbb{C}$ the following operator can be defined

$$f(A) = \int_{\sigma(A)} f(\lambda)|\lambda\rangle\langle\lambda| \, d\mu(\lambda) \tag{3}$$

which should also be interpreted in the sense of (2). This formula is called the Dirac functional calculus formula.

## 3. FORMALISMS ON HILBERT SPACES

In this section, we summarize the attempts for formalizing the Dirac kets in the von Neumann formulation of Quantum Mechanics based in the concept of Hilbert space.

### 3.1. Spectral Measures

J. von Neumann (von Neumann, 1955) identified the observable $A$ with a *self-adjoint operator* on a separable Hilbert space $\mathcal{H}$ and the spectrum of $A$ as the

Hilbert space spectrum $\sigma(A)$ of the operator $A$.

We denote by $(\cdot, \cdot)_{\mathcal{H}}$ or by $(\cdot, \cdot)$ the inner product on $\mathcal{H}$.

– By the *classical spectral theorem*, each self adjoint operator $A$ in $\mathcal{H}$ is associated to a *projection-valued measure or spectral measure* of the form $(\sigma(A), \beta, \mathcal{H}, P)$. We have then

$$A = \int_{\sigma(A)} \lambda \, dP(\lambda),$$

that should be interpreted in the sense that for suitable vectors $f, h, \in \mathcal{H}$ ($h$ in the domain of $A$) one has

$$(f, Ah) = \int_{\sigma(A)} \lambda \, d(f, P(\lambda)h).$$

The same is valid for every measurable function of A in the functional calculus derived.

– In the von Neumann formulation of Quantum Mechanics in Hilbert space the above Dirac formulas only make sense if the spectrum of $A$ is purely discrete (as it happens in the harmonic oscillator for example). In this case, if $\sigma(A) = \{\lambda_k\}$ and $A_k f_k = \lambda_k f_k$ for $k = 1, 2, \ldots,$ we have that, instead of (1) and (2),

$$f = \sum_k (f, f_k) f_k; \quad Af = \sum_k \lambda_k (f, f_k) f_k \tag{4}$$

However, most self adjoint operators representing quantum observables have a continuous spectrum for which (4) is not valid. Therefore, *the Dirac formalism cannot be implemented in the Hilbert space* $\mathcal{H}$.

### 3.2. Direct Integrals of Hilbert Spaces

A possible way out was proposed by Marlow (1965) using direct integrals of Hilbert spaces

$$\mathcal{H}_{\mu,N} = \int_{\sigma(A)} \mathcal{H}_\lambda \, d\mu(\lambda).$$

where,

– The elements of $\mathcal{H}_{\mu,N}$ are $\mu$-measurable fields $\hat{f} := \{\hat{f}(\lambda)\}_{\lambda \in \sigma(A)}$ with $\hat{f}(\lambda) \in \mathcal{H}_\lambda$ such that $\int_{\sigma(a)} \| \hat{f}(\lambda)\|_\lambda^2 d\mu(\lambda) < \infty$ where $\| \cdot \|_\lambda$ denote the norm in $\mathcal{H}_\lambda$. (The subindex $N$ denotes the function $N(\lambda) := \dim \mathcal{H}_\lambda$.)

– We recall that if $(\cdot, \cdot)_\lambda$ denotes the inner product on $\mathcal{H}_\lambda$ the direct integral $\mathcal{H}_{\mu,N}$ is a Hilbert space with the inner product

$$(\hat{g}, \hat{h})_{\mathcal{H}_{\mu,N}} = \int_{\sigma(A)} (\hat{g}(\lambda), \hat{h}(\lambda))_\lambda \, d\mu(\lambda)$$

In addition, if $\{e_k(\lambda)\}_{k=1}^{N(\lambda)}$ is an orthonormal basis of $\mathcal{H}_\lambda$, we have that

$$(\hat{g}, \hat{h})_{\mathcal{H}_{\mu,N}} = \int_{\sigma(A)} \sum_{k=1}^{N(\lambda)} (\hat{g}(\lambda), e_k(\lambda))_\lambda (e_k(\lambda), \hat{h}(\lambda))_\lambda \, d\mu(\lambda) \tag{5}$$

–By the *functional spectral theorem* of von Neumann, for each self adjoint operator $A$ in a Hilbert space $\mathcal{H}$, there exists a Borel measure $\mu$ on $\sigma(A)$, a direct integral of Hilbert spaces $\mathcal{H}_{\mu,N}$ and a unitary operator $V$:

$$V : \mathcal{H} \mapsto \mathcal{H}_{\mu,N}$$

such that the operator $VAV^{-1}$ admits the following integral diagonal form

$$VAV^{-1} = \int_{\sigma(A)} \lambda I_\lambda \, d\mu(\lambda) \tag{6}$$

where $I_\lambda$ is the identity on $\mathcal{H}_\lambda$.
Then, if $f := V^{-1}\hat{f}$, $h := V^{-1}\hat{h}$, (5) and (6) give

$$\begin{aligned}(f, Ah)_{\mathcal{H}} &= (\hat{f}, VAV^{-1}\hat{h})_{\mathcal{H}_{\mu,N}} \\ &= \int_{\sigma(A)} \sum_{k=1}^{N(\lambda)} \lambda (\hat{g}(\lambda), e_k(\lambda))_\lambda (e_k(\lambda), \hat{h}(\lambda))_\lambda \, d\mu(\lambda) \end{aligned} \tag{7}$$

Formula (7) is quite similar to (2), except that in (7) we have considered the possibility of degeneracy of the $\lambda$. If $\phi(\lambda)$ is a given measurable function on $\sigma(A)$, we have the following functional calculus formula:

$$(f, \phi(A)h) = \int_{\sigma(A)} \sum_{k=1}^{N(\lambda)} \phi(\lambda)(\hat{g}(\lambda), e_k(\lambda))_\lambda (e_k(\lambda), \hat{h}(\lambda))_\lambda \, d\mu(\lambda) \tag{8}$$

where the operator $\phi(A)$ is defined on a certain domain $\mathcal{D}_\phi \subset \mathcal{H}$. Formula (8) is sometimes written as:

$$(\phi(A)) = \int_{\sigma(A)} \sum_{k=1}^{N(\lambda)} \phi(\lambda) |e_k(\lambda))(e_k(\lambda)| \, d\mu(\lambda) \tag{9}$$

Compare (9) to (3) omitting the sum in $k$ in (9) for simplicity. Both formulas are identical if we write

$$|e(\lambda)) = |\lambda\rangle.$$

However, the Dirac requirement that each observable should have a complete set of eigenvectors is not fulfilled. If $\lambda \in \sigma(A)$ is in the continuous spectrum of $A$, the Hilbert space $\mathcal{H}_\lambda$ is not a subspace of the direct integral $\mathcal{H}_{\mu,N} = \int_{\sigma(A)} \mathcal{H}_\lambda \, d\mu(\lambda)$, and therefore, we cannot find on $\mathcal{H}_{\mu,N}$ an orthogonal basis of eigenvectors of $A$ (or rather of $VAV^{-1}$). From this point of view, the most we can do, is the following: If $P_\lambda : \mathcal{H}_{\mu,N} \to \mathcal{H}_\lambda$ denote the projection operator, $f \in \mathcal{H}$ is in the domain of the observable $A$ and $Vf = \int_{\sigma(A)} f(\lambda)\, d\mu(\lambda)$ then, it is clear that

$$P_\lambda VAf = \lambda f(\lambda),$$

so that every element of $\mathcal{H}_\lambda$ would be an eigenvector of $A$ in some generalized sense, a.e. in $\mu(\lambda)$. However, if $\lambda$ is in the continuous spectrum of $A$, then, $P_\lambda$ is not even continuous.

## 4. RIGGED HILBERT SPACES

The next attempt to implement the Dirac formulation of Quantum Mechanics will look for these eigenvalues on certain extensions of the Hilbert space of states $\mathcal{H}$ (Foias, 1959a,b, 1962; Gelfand and Vilenkin, 1964; Gelfand and Shilov, 1968; Bohm, 1967; Roberts, 1966; Melsheimer 1974). These are called *rigged Hilbert spaces or Gelfand triplets* and can be defined in four steps

– We start with a topological vector space (tvs) $(\Phi, \tau_\Phi)$, where $\Phi$ denotes a complex vector space and $\tau_\Phi$ a locally convex topology on $\Phi$ (Horváth, 1966; Jarchow, 1981; Schaeffer, 1997).

– Let us consider the space $\Phi^\times$ of continuous *antilinear* mappings (functionals) from $\Phi$ into $\mathbb{C}$. From now on, we shall denote the action of $F \in \Phi^\times$ into $\phi \in \Phi$ as $\langle \phi | F \rangle$. This action is linear to the right and antilinear to the left, just as the scalar product of Hilbert spaces. We shall define $\langle F | \varphi \rangle := \langle \varphi | F \rangle^*$.

– Let us consider the dual pair $(\Phi, \Phi^\times)$ and assume that $\Phi$ is a proper subspace[4] of $\Phi^\times$. If $\varphi \in \Phi$, its action on each arbitrary $\phi \in \Phi$, as an element of the antidual $\Phi^\times$, is given by $\langle \phi | \varphi \rangle$. This is a sesquilinear form on $\Phi$. In addition, if this sesquilinear form is *positive definite* it endows $\Phi$ with the norm $\|\phi\| := \sqrt{\langle \phi | \phi \rangle}$. Its closure with respect to the topology given by this norm is a Hilbert space $\mathcal{H}$. We shall denote the scalar product of two vectors $\varphi, \phi \in \mathcal{H}$ as $(\varphi, \phi)$.

– In the case that the norm $\| \cdot \|$ be one of the seminorms that define the topology $\tau_\Phi$, then

$$\Phi \subset \mathcal{H} \subset \Phi^\times \tag{10}$$

Here, the mapping $\Phi \mapsto \mathcal{H}$ is continuous. A structure like (10) is called a *rigged Hilbert space or Gelfand triplet*. Usually, one demands that, in addition, the canonical mapping $I$ be nuclear. We are going to use this requirement in this section only.

[4] This means that $\Phi$ is algebraically isomorphic to a proper subspace of $\Phi^\times$ that we shall identify with $\Phi$.

### 4.1. The Adjoint Operator

Let A : $\Phi \mapsto \mathcal{H}$ be a continuous operator (with the topologies $(\tau_\phi, \tau_{\mathcal{H}})$). The adjoint operator, $A^\times$, of $A$, $A^\times : \mathcal{H} \to \Phi^\times$ is defined by the relation:

$$(A\phi, f) = \langle \phi | A^\times f \rangle, \quad \forall f \in \mathcal{H}, \forall \phi \in \Phi,$$

The operator $A^\times$ is well defined and is weakly continous—and then strongly continous.

Let $\mathcal{L}(\Phi)$ be the space of continuous linear operators on $\Phi$.
Let $A \in \mathcal{L}(\Phi)$. $A$ has a conjugate $A^c$, if there exists a $A^c \in \mathcal{L}(\Phi)$ such that $(\varphi, A\phi) = (A^c\varphi, \phi)$, for all $\varphi, \phi \in \Phi$.
The space of operators having a conjugate is denoted by $\mathcal{L}^c(\Phi)$.
The operator $A \in \mathcal{L}^c(\Phi)$ is real if $A = A^c$.
For any $A \in \mathcal{L}^c(\Phi)$, we may consider its adjoint

$$A^\times : \Phi^\times \to \Phi^\times$$

which is weakly (and also strongly) continuous on $\Phi^\times$. For any real operator $A$, its adjoint $A^\times$ extends $A$ because

$$\langle \phi | A^\times I^\times I \varphi \rangle = (IA\phi, I\varphi) = (I\phi, IA^c\varphi) = \langle \phi | I^\times I A^c \varphi \rangle, \forall \phi, \varphi \in \Phi,$$

where $I$ is the canonical injection of $\Phi$ into $\mathcal{H}$, i.e., $I(\varphi) = \varphi$, for all $\varphi \in \Phi$. Therefore, $A^\times I^\times I = I^\times I A^c$. Analogously, we have that $A^{c\times} I^\times I = I^\times I A$. Therefore, $A^{c\times}$ is an extension of $A$, which is weakly and strongly continuous.

### 4.2. Integral Decompositions

Let us denote by $\Phi^\times_\beta$ the tvs $\Phi^\times$ with the strong topology $\beta(\Phi^\times, \Phi)$ (see for instance Jarchow, 1981), with respect to the antidual pair $(\Phi, \Phi^\times)$ and let $\gamma$ be an arbitrary mapping in $\mathcal{L}(\Phi, \Phi^\times_\beta)$.
$\gamma$ is *self adjoint* if $\langle \varphi | \gamma \phi \rangle = \langle \phi | \gamma \varphi \rangle^*$ for any pair $\phi, \varphi \in \Phi$.
$\gamma$ is *positive* if $\langle \phi | \gamma \phi \rangle > 0, \forall \phi \in \Phi$.
$\gamma$ is *real* if $\langle \phi | \gamma \phi \rangle \in \mathbb{R}, \forall \phi \in \Phi$.

An *integral decomposition* of $\Phi$ is a triplet $(\Lambda, \mu, \gamma(\lambda))$, with the following properties:
i. The set $\Lambda$ is a locally compact Hausdorf topological space and $\mu$ a regular positive measure on the Borel $\sigma$-algebra $\mathcal{B}$ of $\Lambda$.
ii. $\gamma(\lambda) \in \mathcal{L}(\Phi, \Phi^\times_\beta)$ and $\gamma(A)$ is positive for almost all $\lambda$ with respect to $\mu$.
iii. For each $\varphi, \phi \in \Phi$ the function $\lambda \mapsto \langle \phi | \gamma(\lambda) \varphi \rangle$ is $\mu$ integrable and

$$(\phi, \varphi) = \int_\Lambda \langle \phi | \gamma(\lambda) \varphi \rangle \, d\mu(\lambda).$$

Let $A \in \mathcal{L}^c(\Phi)$.

A functional $\varphi^\times \in \Phi^\times$ is an *eigenform* of A with eigenvalue of $\lambda \in \mathbb{C}$ if

$$\langle A^c\phi|\varphi^\times\rangle = \langle\phi|A^{c\times}\varphi^\times\rangle = \lambda\langle\phi|\varphi^\times\rangle, \quad \forall\phi \in \Phi,$$

This means that the extension, $A^{c\times}$, of $A$ into $\Phi^\times$ fulfills the relation $A^{c\times}\varphi^\times = \lambda\varphi^\times$.
A positive operator $\gamma \in \mathcal{L}(\Phi, \Phi^\times)$ is an *eigenoperator* of $A$ with eigenvalue $\lambda$ if

$$A^{c\times}\gamma = \gamma A = \lambda\gamma.$$

If $\gamma$ is an eigenoperator of $A$, then it is straighforward that $A^\times\gamma = \gamma A^c = \lambda^*\gamma$. Also, if $\gamma$ is an eigenoperator of $A$ with eigenvalue $\lambda$ and if for a given $\varphi \in \Phi$ we have that $\gamma\varphi \neq 0$ then the functional $\varphi^\times = \gamma\varphi$ is an eigenform of $A$ with eigenvalue $\lambda$.
Analogously, $\varphi^\times = \gamma\varphi$ is also an eigenform of $A^c$ with eigenvalue $\lambda^*$.

Being given $A \in \mathcal{L}^c(\Phi)$, we say that $(\Lambda, \mu, \gamma(\lambda))$ is an *A-integral decomposition* of $\Phi$ if for each $\lambda \in \Lambda$, the operator $\gamma(\lambda)$ is either the zero operator or an eigenoperator of $A$. This $A$-decomposition is real if $\Lambda \subseteq \mathbb{R}$.

A closed operator $A$ on $\mathcal{H}$ is *formally normal* if $D(A) \subset D(A^\dagger)$ and $\|Ah\| = \|A^\dagger h\| \forall h \in D(A)$. The operator $A$ is *subnormal* if there exists a normal extension $\tilde{A}$ of $A$ on a Hilbert space $\tilde{\mathcal{H}}$ that contains to $\mathcal{H}$.

In his paper, Roberts (1966a,b) has obtained the following result:

**Proposition 1.** *Let $A \in \mathcal{L}(\Phi)$ and let $\Phi$ be a dense subspace of $\mathcal{H}$ with a locally convex topology such that the canonical injection $\underline{I} : \Phi \to \mathcal{H}$ is continuous. If $\Phi$ has an A-integral decomposition, then the closure $\bar{A}$ of $A$ is a subnormal operator as well as formally normal on $\mathcal{H}$.*

The reciprocal of this result was also proven by Roberts (1966a,b), using the following nuclear spectral theorem of Gårdin and Maurin (Gårdin, 1947; Maurin, 1968):

**Theorem 2.** *Let $\mathcal{H}$ be a Hilbert space and let $\Phi \subset \mathcal{H}$ be a locally convex topological vector space which is nuclear and dense in $\mathcal{H}$, being the canonical injection $I : \Phi \mapsto \mathcal{H}$ continuous. Let $\int_\Lambda \mathcal{H}_\lambda \, d\mu(\lambda)$ be a decomposition of $\mathcal{H}$ as a direct integral of Hilbert spaces. Then, there exists $\mu$-almost everywhere on $\Lambda$ nuclear mappings $I(\lambda) : \Phi \mapsto \mathcal{H}_\lambda$, such that*

$$(I\phi, h) = \int_\Lambda (I(\lambda)\phi, \hat{h}(\lambda))_\lambda \, d\mu(\lambda), \quad \phi \in \Phi, h \in \mathcal{H}. \tag{11}$$

Then, the reciprocal of the last proposition can be presented as follows:

**Proposition 3.** *Let $\mathcal{H}$ and $\Phi$ be as in the previous theorem. Let $A \in \mathcal{L}^{c}(\Phi)$ such that the closure of $IAI^{-1}$ on $\mathcal{H}$ is subnormal and formally normal. Then, $\Phi$ admits an A-integral decomposition with the following eigenoperators:*

$$\gamma(\lambda) = I^{\times}(\lambda)I(\lambda),$$

*where the mappings $I(\lambda)$ fulfill the relation (11) for each direct integral of Hilbert spaces associated to any of the normal extensions of $\overline{A}$ by means of the spectral theorem. Moreover, the A-decomposition of $\Phi$ is real if and only if A is real.*

Two integral decompositions $(\Lambda, \mu_1, \gamma_1(\lambda))$ and $(\Lambda, \mu_2, \gamma_2(\lambda))$ are equivalent if the measures $\mu_1$ and $\mu_2$ are equivalent (belong to the same type $[\mu_1] = [\mu_2]$) and, save for a set of zero measure, we have that

$$\gamma_1(\lambda) = \frac{d\mu_2}{d\mu_1}(\lambda)\gamma_2(\lambda),$$

where $\frac{d\mu_2}{d\mu_1}$ is the Radon–Nikodym derivative of $\mu_2$ with respect to $\mu_1$.

**Proposition 4.** *A real operator $A \in \mathcal{L}^{c}(\Phi)$ has an A-decomposition unique save for equivalence if and only if A is essentially self adjoint on $\mathcal{H}$. In this case, if P is the spectral measure provided by the spectral representation of $\overline{A}$, the closure of A, and $(\mathbb{R}, \mu, \gamma)$ is an A-decomposition of $\Phi$, we have that*

$$(\phi, P(E)\varphi) = \int_E \langle\phi|\gamma(\lambda)\varphi\rangle\, d\mu(\lambda) = \int_E (I(\lambda)\phi, I(\lambda)\varphi)_\lambda\, d\mu(\lambda),\ \phi, \varphi \in \Phi.$$

### 4.3. Representation of the Eigenoperators

If $\Phi \subset \mathcal{H} \subset \Phi^{\times}$ is a rigged Hilbert space for which the space $\Phi$ is nuclear and separable, O. Melsheimer (Melsheimer, 1974) has found a representation of the eigenoperators of an $A \in \mathcal{L}^{c}(\Phi)$, essentially self adjoint on $\mathcal{H}$, in terms of the eigenforms of the adjoint of $A$, $A^{\times}$. We present this representation as follows:

Let $(\mathbb{R}, \mathcal{B}, \mathcal{H}, P)$ be the spectral measure of the closure $\overline{A}$ of $A$.

For each $g \in \mathcal{H}$, $\mathcal{H}_g$ is the closed subspace of $\mathcal{H}$ spanned by the vectors of the form $P(E)g$, with $E \in \mathcal{B}$. The orthogonal projection $P_g$ of $\mathcal{H}$ on $\mathcal{H}_g$ commutes with $P(E)$ for all $E \in \mathcal{B}$.

Then, there exists a unique ($\mu$ almost everywhere) real functional $\phi_g^{\times}(\lambda) \in \Phi^{\times}$ (i.e., a functional from $\Phi$ onto $\mathbb{R}$) such that (Melsheimer, 1974)

$$(I_\phi, P_g P(E) I\varphi) = \int_E \langle\phi|\phi_g^{\times}(\lambda)\rangle\langle\varphi|\phi_g^{\times}(\lambda)\rangle^*\, d\mu(\lambda),$$

for each $\phi, \varphi \in \Phi$ and $E \in \mathcal{B}$. The expression that Melsheimer obtains for $\phi_g^\times(\lambda)$ is the following:

$$\phi_g^\times(\lambda) = \frac{I^\times(\lambda)\hat{g}(\lambda)}{\hat{g}(\lambda)}, \tag{12}$$

where $\tilde{g}$ is the image of $g$ by the isomorphism $\mathcal{H}_g \sim L^2_\mu$.

In general if $\{g_n\}$ is a complete sequence of generators of $\mathcal{H}$, i.e., $\mathcal{H} = \oplus \mathcal{H}_{g_n}$ then the sequence $\{\gamma_{gn}(\lambda)\}$ converges absolutely ($\mu$ a.e.) to a $\gamma(\lambda)$ in the strong topology on $\mathcal{L}(\Phi, \Phi_\beta^\times)$. Furthermore, $\gamma(\lambda)\phi$ admits the following representation:

$$\gamma(\lambda)\phi = \sum_n \langle \phi | \phi_{gn}^\times(\lambda)\rangle^* \phi_{gn}^\times(\lambda).$$

In Melsheimer (Melsheimer, 1974), the author cannot go further because the explicit form of the mappings $I(\lambda)$ and $I^\times(\lambda)$, in terms of the operator $A$ or the associated spectral measure, were unknown. In Melsheimer (Melsheimer, 1974), the existence of these two mappings was derived from the nuclear spectral theorem. The proof of this theorem is based on the nuclearity of the canonical injection I $I : \Phi \mapsto \mathcal{H}$, which has a representation of the form $I(\cdot) = \sum_k \lambda_k \langle \cdot | \varphi_k^\times \rangle h_k$[5] where the knowledge of the explicit form of the sequences $\{h_k\}$, $\{\varphi_k\}$, and $\{\lambda_k\}$ is not necessary. We concrete these expressions below.

## 5. EIGENFUNCTION EXPANSIONS OF KATO–KURODA

Retjo (1967), Howland (1967, 1968, 1986), Kuroda (1967) and Kato (1970) constructed a theory of *eigenfunction expansions* in which, as in the Gelfand and Foias theories, such eigenfunctions—the generalized eigenvectors—belong to the antidual space $\Phi^\times$ of an auxiliary tvs $\Phi$. In this theory, $\Phi$ need not be a dense subset of $\mathcal{H}$ and the eigenfunction expansion is formulated in an abstract way analogous to that given by Gelfand and Shilov (1968) and Gelfand and Vilenkin (1964).

### 5.1. Spectral Forms

By a *spectral system* $(\Lambda, \mathcal{A}, \mu, \mathcal{H}, P)$ we mean a spectral measure space $(\Lambda, \mathcal{A}, \mathcal{H}, P)$ together with a $\sigma$-finite nonnegative scalar measure $\mu$ on $(\Lambda, \mathcal{A})$.

By a standard process, $P$ is decomposed into the absolutely continuous part $P^{\mathrm{ac}}$ and the singular part $P^{\mathrm{s}}$ with respect to $\mu$ (Birman and Solomjak, 1987). The basic elements of the Kato–Kuroda theory are the *spectral forms* for such systems:

*Definition*. Let $(\Lambda, \mathcal{A}, \mu, \mathcal{H}, P)$ be a spectral system. *A spectral form* for this system is a complex function on

[5] Here, $\{h_k\}$ is a bounded sequence in $\mathcal{H}$, $\{\varphi_k\}$ an equicontinuous sequence in the antidual $\Phi^\times$ and $\{\lambda_k\}$ a sequence of complex numbers with $\sum |\lambda_k| < \infty$.

$\hat{\Lambda} \times \Phi \times \Phi$ where $\Phi$ is a subspace of $\mathcal{H}$ and $\hat{\Lambda} \subseteq \Lambda$ belongs to $\mathcal{A}$, with the following properties.

i. For each $\phi, \varphi \in \Phi$, $\lambda \mapsto s(\lambda; \varphi, \phi)$ is $\mu$-integrable in $\hat{\Lambda}$ and its integral on each $E \subseteq \hat{\Lambda}$, $E \in \mathcal{A}$, is equal to $(\varphi, P^{ac}\phi)$, i.e.,

$$(\varphi, P^{ac}(E)\phi) = \int_E s(\lambda; \varphi, \phi)\, d\mu(\lambda) \tag{13}$$

ii. For each $\lambda \in \hat{\Lambda}$ the function $\varphi, \phi \mapsto s(\lambda; \varphi, \phi)$ is a nonnegative Hermitian form on $\Phi \times \Phi$. (We write $s(\lambda; \phi)$ for $s(\lambda; \phi, \phi)$.)

The subspace $\Phi$ is usually called a *spectral subspace* and the subset $\hat{\Lambda}$ a *spectral core* of the spectral system. The spectral from is denoted by $(\hat{\Lambda}, \Phi, s)$.

Since (f, $P^{ac}(\cdot)h$) is a complex-valued $\mu$-absolutely continuous measure for each $f, h \in \mathcal{H}$ the Radon–Nikodym derivative $\frac{d(f, P(\lambda)h)}{d\mu} = \frac{d\mu_{f,h}}{d\mu}(\lambda)$ is defined for $\mu$-a.e. $\lambda \in \lambda$ and by (13) we have

$$s(\lambda;\, f, h) = \frac{d\mu_{f,h}}{d\mu}(\lambda),\ \mu\text{-a.e.} \tag{14}$$

The null set depending on $f$ and $h$, it is in general difficult to choose $s(\lambda;\, f, h)$ as an Hermitian form in $\mathcal{H}$ for each $\lambda \in \Lambda$. But it can be done easily if f, h, and $\lambda$ are suitably restricted. This is what the last definition is concerned with. (When the spectral forms $s$ is that of (14) we refer to s as the *canonical spectral form*.)

*Example.* Let $\mathcal{H} = L^2(\mathbb{R})$, $\Lambda = \mathbb{R}$ with the Lebesgue measure $d\lambda$ and let $P(E)$ be the operator of multiplication by the characteristic function of $E$ defined on the Borel $\sigma$-algebra $\mathcal{B}$ of $\mathbb{R}$. If $\Phi$ is the set of all continuous functions in $L^2(\mathbb{R})$, then $s(\lambda; \varphi\phi) = \varphi^*(\lambda)\phi(\lambda)$ defines a spectral form on $\mathbb{R} \times \Phi \times \Phi$, called the canonical spectral form.

*Definition 5.* Let $(\Lambda, \mathcal{A}, \mu, \mathcal{H}, P)$ be a spectral system with a spectral form $(\hat{\Lambda}, \phi, s)$. A function

$$\begin{array}{ll} (0, \epsilon_0) \times \mathbb{R} \times \Phi \times \Phi \to \mathbb{C} \\ \quad (\epsilon, \lambda, \varphi, \phi) \qquad \mapsto \mathcal{S}_\epsilon(\lambda; \varphi, \phi) \end{array}, \quad (\epsilon_0 > 0).$$

is called an *approximate spectral form* for $(\hat{\Lambda}, \Phi, s)$ if for each $\varphi, \phi \in \Phi$ and $\lambda \in \hat{\Lambda}$, $\mathcal{S}_\epsilon(\lambda; \varphi, \phi)$ is a Cauchy net when $\epsilon \to 0$ and its limit is equal to

$$\lim_{\epsilon \to 0} \mathcal{S}_\epsilon(\lambda; \varphi, \phi) = \frac{d\mu_{\varphi,\phi}}{d\mu}(\lambda).$$

*Example.* Let $\mathcal{H} = L^2(\mathbb{R})$, $\mu$ the Lebesgue measure, and let $P(E)$ be the multiplication by the characteristic function $\chi_E$, $E \in \mathcal{B}$. If $z \in \mathbb{C}\backslash\mathbb{R}$, let $R(z) := J_{(\lambda - z)^{-1}}$

be the resolvent[6] of $P$ in $z$. The following result for the *Poisson integral* is well known (Baumgärtel and Wollenberg, 1983; prop. 3.16): For any $f \in \mathcal{H}$ and for any $\lambda \in \mathbb{R}$ where $d\mu_f(\lambda)/d\lambda$ exists, we have

$$\frac{d(f, P(\lambda) f)}{d\lambda} = \lim_{\epsilon\to 0} \frac{\epsilon}{\pi}(R(\lambda + i\epsilon) f, R(\lambda + i\epsilon) f)$$
$$= \lim_{\epsilon\to 0} \frac{\epsilon}{\pi} \int \frac{1}{(\lambda - x)^2 + \epsilon^2}\, d\mu_f(x).$$

Let us consider the spectral form $(\mathbb{R}, \Phi, s)$, where $\Phi$ is the space of all continuous functions of $L^2(\mathbb{R})$ and $s(\lambda; \varphi, \phi) = \varphi^*(\lambda)\phi(\lambda)$ is the canonical spectral form. Then an approximate spectral form for $(\mathbb{R}, \Phi, s)$ is given by

$$r_\epsilon(\lambda; \varphi, \phi) := (\epsilon/\pi)(R(\lambda + i\epsilon)\varphi, R(\lambda + i)\epsilon\phi). \tag{15}$$

Finally, we assume that the spectral subspace $\Phi$ is a topological vector space (tvs) with its own topology. The following result is obvious.

**Proposition 6.** *Let $(\hat\Lambda, \Phi, s)$ be a spectral form and let $s_\epsilon$ an approximate spectral form for it which is equicontinuous with respect to $\epsilon > 0$ for each $\lambda \in \hat\Lambda$. Then the spectral form $s(\lambda, \cdot, \cdot)$ is continous in $\Phi \times \Phi$ for each $\lambda \in \hat\Lambda$.*

*In the example considered above the approximate spectral form $\tau_\epsilon(\lambda; \varphi, \phi)$ is equicontinuous with respect to $\epsilon > 0$ if the topology in $\Phi$ is defined by $\|\phi\|_\Phi :=$* $\sup_{\lambda\in\mathbb{R}} |\phi(\lambda)|$ *In this case we have*

$$\frac{\epsilon}{\pi}\|R(\lambda + i\epsilon)\phi\|^2 = \frac{\epsilon}{\pi}\int_{-\infty}^{\infty} \frac{|\phi(u)|^2}{(\lambda - u)^2 + \epsilon^2}\, du \le \|\phi\|_\Phi^2.$$

## 5.2. Spectral Representations

Let $(\Lambda, \mathcal{A}, \mu, \mathcal{H}, P)$ be a spectral system with a spectral form $(\hat\Lambda, \Phi, s)$. For each $\lambda \in \hat\Lambda$, $s(\lambda; \cdot, \cdot)$ defines a semi-inner product in $\Phi$. Let $\mathcal{N}_\lambda$ the set of all $\phi$ with $s(\lambda; \phi) = 0$, $s(\lambda, \phi) := s(\lambda, \phi, \phi)$. Then, the quotient space $\Phi/\mathcal{N}_\lambda$ is a pre-Hilbert space with the inner product induced by $s(\lambda; \cdot, \cdot)$. We denote by $\tilde\Phi_\lambda$ its completion, by $(\cdot, \cdot)_\lambda$ and $\|\cdot\|_\lambda$ the inner product and norm in $\tilde\Phi_\lambda$ and by $q_\lambda$ the quotient map of $\Phi$ onto $\Phi/\mathcal{N}_\lambda \subset \tilde\Phi_\lambda$.

Let us consider the product vector space $\tilde\Phi_\lambda = \prod_{\lambda\in\hat\Lambda} \tilde\Phi_\lambda$ consisting of all vector fields $\tilde\phi = \{\tilde\phi(\lambda)\}_{\lambda\in\hat\Lambda}$ with $\tilde\phi(\lambda) \in \tilde\Phi_\lambda$, where we identify the elements equal $\mu$-a.e.

[6] We define

$$J_{f(x)} := \int_\Lambda f(\lambda)\, dP(\lambda).$$

By a *quasi-simple function* $\tilde{\phi}$ we mean a function of the form (finite sum)

$$\tilde{\phi}(\lambda) = \sum \alpha_k(\lambda)\phi_k, \quad \alpha_k \in L^{\infty}(\mu), \phi_k \in \Phi.$$

*Definition 7.* $A\,\tilde{\phi} \in \tilde{\Phi}$ is said to be *s-measurable function* if there is a sequence $\{\tilde{\phi}_k\}$ of quasi-simple functions on $\hat{\Lambda}$ such that

$$\lim_{n\to\infty} \|\tilde{\phi}(\lambda) - q_\lambda \phi_n(\lambda)\|_\lambda = 0 \quad \text{for} \quad \mu\text{-a.e.}\lambda \in \hat{\Lambda}.$$

We denote by $\mathcal{H}_{\mu,\Phi}$ the set of all $s$-measurable elements $\tilde{\phi} \in \tilde{\Phi}$ such that $\|\tilde{\phi}\|^2_{\mathcal{H}_{\mu,\Phi}} = \int_{\hat{\Lambda}} \|\tilde{\phi}(\lambda)^2_\lambda\, d\mu(\lambda) < \infty$. Thus $\mathcal{H}_{\mu,\Phi}$ is a Hilbert space with the inner product

$$(\tilde{\varphi}, \tilde{\phi})_{\mathcal{H}_{\mu,\Phi}} := \int_{\hat{\Lambda}} (\tilde{\varphi}(\lambda), \tilde{\phi}(\lambda))_\lambda\, d\mu(\lambda) < \infty.$$

**Proposition 8.** *Quasi-simple functions on $\hat{\Lambda}$ to $\Phi$ are densely embedded in $\mathcal{H}_{\mu,\Phi}$ in the following sense:*

*(a) For any quasi-simple function $\tilde{\phi}$ we have $q\tilde{\phi} := \{q\lambda\tilde{\phi}(\lambda)\} \in \mathcal{H}_{\mu,\Phi}$.*
*(b) For each $\tilde{\phi} \in \mathcal{H}_{\mu,\Phi}$ and $\epsilon > 0$, there is a quasi-simple function $\tilde{\varphi}$ such that $\|q\tilde{\varphi} - \phi\|_{\mathcal{H}_{\mu,\Phi}} < \epsilon$.*

We denote by $\mathcal{H}_\Phi$ the smallest closed subspace of $\mathcal{H}$ containing $\Phi$ and reducing $P$, i.e., $P\mathcal{H}_\Phi$ Note that $\mathcal{H}_\Phi = \mathcal{H}$ if $\Phi$ is dense. $\mathcal{H}_\Phi$ is the closed span of the set of all vectors of the form $[\int_{\hat{\Lambda}} \alpha(\lambda)\, dP(\lambda)]\phi$ with $\alpha \in L^{\infty}(\mu)$ and $\phi \in \Phi$.

Hence $\mathcal{H}^{\rm ac}_\Phi := P^{\rm ac}\mathcal{H}_\Phi$ also reduces $P$ and $\mathcal{H}^{\rm ac}_\Phi := \mathcal{H}^{\rm ac}$ if $\Phi$ generates $\mathcal{H}$ (i.e if $\mathcal{H}_\Phi := \mathcal{H}$).

The following theorem is the main result of this section (Kato and Kuroda, 1970; theorem 1.11):

**Theorem 9.** *There is a unitary operator $V$ on $\mathcal{H}^{\rm ac}_\Phi$ into the direct integral $\mathcal{H}_{\mu,\Phi}$ with the following properties:*

*1. $V J_\alpha(P)h = \alpha V h = \{\alpha(\lambda) V h(\lambda)\}_{\lambda\in\hat{\Lambda}}$ for each $\alpha \in L^{\infty}(\mu)$ and $h \in \mathcal{H}^{\rm ac}$. where $J_\alpha(P) := \int_{\hat{\Lambda}} \alpha(\lambda)\, dP(\lambda)$.*
*2. $V P^{\rm ac}(\hat{\Lambda}) = \{q_\lambda \phi\}_{\lambda\in\hat{\Lambda}}$ for each $\phi \in \Phi$.*

The relation between the operator V here and the operator V that appeared in section 3.2 will be clarified in Theorem 19.

### 5.3. Eigenfunction Expansions

Let $(\Lambda, \mathcal{A}, \mu, \mathcal{H}, P)$ be a spectral system with a spectral form $(\hat{\Lambda}, \Phi, s)$.

Kato and Kuroda introduced the following conditions, which imply that the system has a representation in a somewhat more refined sense:

E1) There exists a $\sigma$-finite measure space $(\Gamma, \mathcal{B}, \rho)$, a partial isometry $W$ of $\mathcal{H}$ onto $L^2(\Gamma, \rho)$ with intial set $\mathcal{H}^{\text{ac}}$, and a measurable function $w : \Gamma \to \Lambda$ such that

$$[W J_\alpha h](\xi) = \alpha(w(\xi))[Wu](\xi), \quad \rho\text{-a.e.}\xi \in \Gamma, \tag{16}$$

for each $h \in \mathcal{H}$ and $\alpha \in L^\infty(\Lambda, \mu)$. (The measurability of $w$ means that $w^{-1}(E) \in \mathcal{B}$ whenever $E \in \mathcal{A}$. Thus $\alpha \circ w$ is $\rho$-measurable on $\Gamma$ if $\alpha$ is $\mathcal{A}$-measurable on $\Lambda$.)

E2) There is a complex-valued function $\psi$ on $\Gamma \times \Phi$ such that for each fixed $\xi \in \Gamma$, $\phi \mapsto \psi(\xi; \phi)$ is linear and for each fixed $\phi \in \Phi$,

$$\psi(\xi; \phi) = [W\phi](\xi), \quad \rho\text{-a.e.}\xi \in \Gamma. \tag{17}$$

E3) $\Phi$ is a topological vector space and $\phi \mapsto \psi(\xi; \phi)$ is continuous on $\Phi$ for each $\xi \in \Gamma$. In this case we write $\psi(\xi; \phi) = \langle \phi | \psi^\times(\xi) \rangle$ where $\psi^\times \in \Phi^\times$. Each $\psi^\times(\xi)$ will be called an *eigenfunction* of $P$.

*Example.* Let $\mathcal{H} = L^2(\mathbb{R}^3)$, $W$ the Fourier–Plancherel transformation of $\mathcal{H}$ onto $\hat{\mathcal{H}} = L^2(\Gamma, \mathcal{B}, \rho)$ where $\Gamma$ is another copy of $\mathbb{R}^3$, $\mathcal{B}$ the Borel $\sigma$-algebra on $\Gamma$, and let $\rho$ be the Lebesgue measure $d\xi$. If map $w$ of $\Gamma$ into $\hat{\Lambda} = \Lambda\mathbb{R}^+$ is given by $w(\xi) = |\xi|^2 = \xi_1^2 + \xi_2^2 + \xi_3^2$ is the Borel $\sigma$-algebra of $\Lambda$, $\mu$ the Lebesgue measure on $\Lambda$, $P(E) = W^{-1}\hat{P}(E)W$, where $\hat{P}(E)$ is the operator of multiplication by $\chi_{\omega^{-1}E}$ (this is the characteristic function of the set $w^{-1}(E)$, the function which is one on the set and zero otherwise), and $\Phi = L^1(\mathbb{R}^3) \cap \mathcal{H}$ with the $L^1$-topology, then $\psi^\times(\xi) \in \Phi^\times$ is given by the function $(2\pi)^{-3/2} e^{i\xi\cdot\lambda}$ in the sense that

$$\langle \phi | \psi^\times(\xi) \rangle = (2\pi)^{-3/2} \int_{\mathbb{R}^3} \phi(\lambda)^* \, e^{i\xi\cdot\lambda}, d\lambda, \phi \in \Phi \tag{18}$$

The spectral measure $P$ is the one associated with the selfadjoint operator—$\Delta$ in $\mathcal{H}$ and the $\phi^\times(\xi)$ are the eigenfunctions of this operator in the usual sense. In this case, a spectral form is

$$s(\lambda; \varphi, \phi) = (4\pi^2)^{-1} \int_{\mathbb{R}^3 \times \mathbb{R}^3} \frac{\sin(\lambda^{1/2}|x - y|)}{|x - y|} \phi(x) \varphi^*(y) \, dx \, dy, \tag{19}$$

where $\lambda \in \hat{\Lambda}$ and $\varphi, \phi \in \Phi$.

## 6. LOCALLY CONVEX EQUIPMENTS OF A SPECTRAL MEASURE

In the formalisms we have described above the fundamental formulas satisfied for the complete systems of generalized eigenvectors are very similar:

For the direct integrals of Hilbert spaces we have

$$(f, P_{(a,b]}h)_{\mathcal{H}} = \int_a^b \sum_{j=1}^{\dim \mathcal{H}_\lambda} (f(\lambda), e_j(\lambda))_\lambda (e_j(\lambda), h(\lambda))_\lambda \, d\mu(\lambda),$$

where $f, g \in \mathcal{H}$ and $(a, b]$ is an arbitray interval of $\mathbb{R}$

For the rigged Hilbert spaces, the mappings $I(\lambda) : \Phi \mapsto \mathcal{H}_\lambda$ of the nuclear spectral theorem, from which we determine the form of the eigenoperators and eigenforms, satisfy

$$(I\phi, P(E)h) = \int_E (I(\lambda)\phi, \hat{h}(\lambda))_\lambda \, d\mu(\lambda),$$

for every $\phi \in \Phi$, $h \in \mathcal{H}$ and $E \subset \Lambda$, $E \in \mathcal{A}$. (Here we consider a spectral measure $(\Lambda, \mathcal{A}, \mathcal{H}, P)$ associated to the direct integral.)

Finally, in the Kato–Kuroda theory, the spectral forms s satisfy

$$(\varphi, P^{\mathrm{ac}}(E)\phi) = \int_E s(\lambda; \varphi, \phi) \, d\mu(\lambda),$$

for each $\phi, \varphi \in \Phi$ and $e \subset \Lambda_1$, $E \in \mathcal{A}$.

Other common feature of the formulations requiring an auxiliar topological vector space $\Phi \subset \Lambda$ is the restriction of the action of the eigenvectors to this subspace $\phi$ and to a subset of the domain $\Lambda$.

These facts motivate the following definition of the equipments of a spectral measure:

*Definition 10.* The topological vector spaces $(\Phi, \tau_\Phi)$ equip the spectral measure $(\Lambda, \mathcal{A}, \mathcal{H}, P)$ if and only if the following conditions hold:

i. There exists a one-to-one linear mapping $I : \Phi \mapsto \mathcal{H}$ with range (image in $\mathcal{H}$ of $\Phi$ by $I$) a dense space in $\mathcal{H}$. If we identify each $\phi \in \Phi$ with its image, $I(\phi)$ in $\mathcal{H}$, we can assume that $\Phi \subset \mathcal{H}$ is a dense subspace of $\mathcal{H}$ and $I$ the canonical injection from $\Phi$ into $\mathcal{H}$.
ii. (First version). There exists a $\sigma$-finite measure $\mu$ on $(\Lambda, \mathcal{A})$, a set $\Lambda_0 \subset \Lambda$ with zero $\mu$ measure and a family of vectors in $\Phi^\times$ of the form

$$\{|\lambda k^\times\rangle \in \Phi^\times : \lambda \in \Lambda \backslash \Lambda_0, k \in [1, m\rangle\}, \tag{20}$$

where $m \in \{\infty, 1, 2, \ldots\}$, such that

$$(\phi, P(E)\varphi)_{\mathcal{H}} = \int_E \sum_{k=1}^{m} \langle \phi|\lambda k^\times\rangle \langle \varphi|\lambda k^\times\rangle^* \, d\mu(\lambda), \quad \forall \phi, \varphi \in \Phi, \forall E \in \mathcal{A}. \tag{21}$$

In particular, if $E = \Lambda$, then, $P(E) = I_{\mathcal{H}}$, the identity on $\mathcal{H}$ and

$$(\phi, \varphi)_{\mathcal{H}} = \int_\Lambda \sum_{k=1}^{m} \langle \phi | \lambda k^\times \rangle \langle \varphi | \lambda k^\times \rangle^* \, d\mu(\lambda), \quad \forall \phi, \varphi \in \Phi.$$

iii. (Second version) There exists a $\sigma$-finite measure on $(\Lambda, \mathcal{A})$, a set $\Lambda_0 \subset \Lambda$ with zero $\mu$ measure and a mapping:

$$\begin{array}{lcl} (\Lambda \backslash \Lambda_0) \times [1, m\rangle & \to & \Phi^\times \\ \lambda \times k & \mapsto & |\lambda k^\times\rangle, \end{array} \tag{22}$$

such that for each $\phi \in \Phi$, the complex function $\langle \phi | \lambda k^\times \rangle$ belongs to $L^2(\Lambda \times [1, m\rangle, \mu \times d)$, where $d$ is the discreet measure on $|1, m\rangle$, i.e.,

$$\int_\Lambda \sum_{k=1}^{m} |\langle \phi | \lambda k^\times \rangle|^2 \, d\mu(\lambda) < \infty, \quad \forall \phi \in \Phi,$$

and relations (21) hold.

Each family of the form (20) or (22) satisfying (21) is called a *complete system of Dirac kets* (also called generalized eigenvalues) of the spectral measure $(\Lambda, \mathcal{A}, \mathcal{H}, P)$ on the rigging $(\Phi, \tau_\Phi)$ and will be denoted by $(\Phi, \tau, \mu, \lambda k^\times)$.

Formula (21) determines the complete system of Dirac kets of the spectral measure uniquely except for:

i. The $\mu$-zero measure set $\Lambda_0$.
ii. The order of the vectors $|\lambda 1^\times\rangle, |\lambda 2^\times\rangle, \ldots, |\lambda N(\lambda)^\times\rangle$.
iii. A faze factor for each $|\lambda k^\times\rangle$. If $|\lambda k^\times\rangle$ is a complete system of Dirac kets, then also is $e^{i\phi(\lambda k)}|\lambda k^\times\rangle$, $\phi$ being a measurable function from $\Lambda \times \mathbb{N}$ into $\mathbb{R}$.

We may assume without loss of generality that $[P] = [\mu]$.

### 6.1. The Action of Eigenvectors

Let $(\Lambda, \mathcal{A}, \mathcal{H}, P)$ be a spectral measure space. In the following we assume that the conditions of Birman–Solomjak are satisfied, i.e., every Hilbert space is separable and every measure space has a numerable basis. These conditions don't suppose any restriction on applications to Quantum Mechanics.

By $\mathcal{H}_g$ we denote the closure of the space

$$\{f \in \mathcal{H} : f = P(E)_g\},$$

for any Borel set $E$ in $\sigma(A)$. We say that the sequence of vectors $\{g_j\}_{j=1}^m$, $m = 1, 2, \ldots, \infty$ in $\mathcal{H}$ is a *generating system* of $\mathcal{H}$ if

$$\mathcal{H} = \bigoplus_{j=1}^{m} \mathcal{H}_{g_j}.$$

If $f, g \in \mathcal{H}$, $\mu_{f,g}(E) := (f, P(E)g)$ when $E \in \sigma(A)$ is a complex measure on $\sigma(A)$. If $f = g$, we write $\mu_f := \mu_{f,f}$ and therefore, $\mu_f(E) = (f, P(E)f)$. The support of $\mu_g$ is $\Lambda(g)$, so that, $\Lambda(g) \subset \sigma(A)$. The *type* of the measure $\mu$ on $\sigma(A)$ is the equivalence class of all measures that are equivalent with $\mu$ and is denoted by $[\mu]$ (two measures, on the same measurable space $\Omega$, $\mu$ and $\nu$ are equivalent if $\mu$ is absolutely continuous with respect to $\nu$ and vice versa). If the measure $\mu$ is absolutely continuous with respect to the measure $\nu$, we write $[\nu] \succ [\mu]$.

A nonzero vector $g \in \mathcal{H}$ is of *maximal type* with respect to the spectral measure $P$ if for each $f \in \mathcal{H}$, $[\mu_g] \succ [\mu_f]$ and g is a maximal vector. Such maximal vectors always exist. The type $[\mu_g]$ of a maximal vector is called the *spectral type* of P and denoted by $[P]$.

Now, let us consider a direct integral of Hilbert spaces associated to the spectral measure space:

$$\mathcal{H}_{\mu,N} = \int_\Lambda \mathcal{H}_\lambda \, d\mu(\lambda)$$

and let $V : \mathcal{H} \mapsto \mathcal{H}_{\mu,N}$ the unitary operator of the functional spectral theorem of von Neumann (1955). Birman and Solomjak (1987) determine the explicit form of $V$ in terms of a particular generating system of the spectral measure and an orthonormal measurable basis of the direct integral:

**Proposition 11.** *There exists a generating system $\{g_k\}_{k=1}^m$ in $\mathcal{H}$, with respect to $P$, such that:*

*i.* $[P] = [g_1] \succ [g_2] \succ \cdots$.

*ii. If $\{e_j(\lambda)\}_{j=1}^m$ is a measurable orthonormal basis on $\mathcal{H}_{\mu,N}$, then, for all $h \in \mathcal{H}_{\mu,N}$, we have that:*

$$V^{-1}h = \bigoplus_{j=1}^{m} \left( \int_{\sigma(A)} \left( e_j(\lambda), \sqrt{\frac{d\mu}{d\mu_{g_1}}(\lambda)} h(\lambda) \right)_\lambda dP(\lambda) \right) g_j \qquad (23)$$

**Theorem 12.** *Under the conditions of Proposition 1, for each $f, h \in \mathcal{H}$ we have*

$$(f, P(E)h) = \sum_{j=1}^{m} \int_E (Vf(\lambda), e_j(\lambda))_\lambda (e_j(\lambda), Vh(\lambda))_\lambda \, d\mu(\lambda) \qquad (24)$$

**Theorem 13.** *Under the same conditions as in Theorem 12, for each $h \in \mathcal{H}$ and $k \in |1, m\rangle$ (see section 2 for a definition of $|1, m\rangle$ we have the following identities:*

$$
\begin{aligned}
(e_k(\lambda), Vh(\lambda))_\lambda &= \sqrt{\frac{d\mu}{d\mu_{g_1}}(\lambda)}\frac{d\mu_{g_k,h}}{d\mu}(\lambda) \\
&= \sqrt{\frac{d\mu_{g_1}}{d\mu}(\lambda)}\frac{d\mu_{g_k,h}}{d\mu_{g_1}}(\lambda) \qquad (25) \\
&= \sqrt{\frac{d\mu_{g_k}}{d\mu}(\lambda)}\frac{d\mu_{g_k,h}}{d\mu_{g_k}}(\lambda).
\end{aligned}
$$

*Example.* Let us consider the position operator on $L^2(\mathbb{R}, dx)$ defined as usual by

$$
\begin{aligned}
Q : \mathcal{D}_Q &\to L^2(\mathbb{R}, dx) \\
f &\mapsto x.f(x)
\end{aligned}
$$

with domain

$$\mathcal{D}_Q = \left\{ f \in L^2(\mathbb{R}, dx) : \int_{\mathbb{R}} |xf(x)|^2\, dx < \infty \right\}.$$

The operator $Q$ is self adjoint on this domain with spectral measure given by

$$P(E)f = \chi_E \cdot f, \quad \forall f \in L^2(\mathbb{R}, dx), \forall E \in \mathcal{B}.$$

where $\mathcal{B}$ is the class of Borel sets in the real line. The spectral measure of $Q$ is given by

$$(\mathbb{R}, \mathcal{B}, L^2(\mathbb{R}, dx), P)$$

For, any $f \in L^2(\mathbb{R}, dx)$ which is a.e. different from zero almost elsewhere with respect to the Lebesgue measure on $\mathbb{R}$ is a generating vector. The direct integral given by the von Neumann theorem

$$\mathcal{H}_{\mu,N} = \int_{\mathbb{R}} \mathcal{H}_x\, d\mu(x)$$

is equal to $L^2(\mathbb{R}, dx)$. Therefore, $N(\lambda) = 1$, $\mathcal{H}_x = \mathbb{C}$ and $d\mu(x) = dx$. The measurable orthonormal basis $e(x)$ in our case are measurable functions $e(x) : \mathbb{R} \mapsto \mathbb{C}$ such that

$$|e(x)| = 1$$

almost everywhere with respect to the Lebesgue measure. The unitary mapping $V$ is given by

$$V^{-1} : L^2(\mathbb{R}, dx) \to L^2(\mathbb{R}, dx)$$

$$Vh \mapsto \left( \int_{\mathbb{R}} e^*(x) \sqrt{\frac{d\mu}{d\mu_g}(x)} Vh(x)\, dP(x) \right) g.$$

Thus,

$$(e(x), Vh(x))_x = \sqrt{\frac{d\mu_g}{d\mu}(x)} \frac{d\mu_{g,h}}{d\mu_g}(x). \tag{26}$$

The measures on (26) are given by

$$\mu_{g,h}(E) = (g, P(E)h) = \int_E g^*(x)h(x)\, dx, \quad \forall E \in \mathcal{B},$$

$$\mu_{g,g}(E) = (g, P(E)g) = \int_E g^*(x)g(x)\, dx, \quad \forall E \in \mathcal{B}.$$

From these identities, we easily obtain the Radon–Nikodym derivatives of the measures $\mu_{h,g}$ and $\mu_g$ with respect to the Lebesgue measure on $\mathbb{R}$. These Radon–Nikodym derivatives are:

$$\frac{d\mu_{g,h}}{dx}(x) = g^*(x)h(x) \quad \text{and} \quad \frac{d\mu_g}{dx}(x) = g^*(x)g(x). \tag{27}$$

As the generating vector $g(x)$ is a.e. different from zero, we can divide the first identity in (27) by the second to obtain

$$\frac{d\mu_{g,h}}{d\mu_g}(x) = \frac{h(x)}{g(x)}. \tag{28}$$

If we replace (28) in (26), we conclude that the action of the measurable orthonormal basis on each $\mathcal{H}_x$ is proportional to the action of the Dirac delta $\delta(x)$:

$$(e(x), \mathcal{F}h(x))_x = \sqrt{\frac{d\mu_g}{d\mu}(x)} \frac{h(x)}{g(x)}, \ \mu\text{-a.e.} \tag{29}$$

In particular, if $\mu = \mu_g$, we have

$$\frac{h(x)}{g(x)} = \frac{1}{g(x)} \delta_q(h),$$

almost elsewhere with respect to $\mu$.

## 6.2. Minimal Riggings

We can define the following partial order in the class of the riggings of a spectral measure:

*Definition 14.* Let $(\Phi, \tau_\Phi)$ and $(\Psi, \tau_\Psi)$ be two riggings of the spectral measure $(\Lambda, \mathcal{A}, \mathcal{H}, P)$, we say that $(\Phi, \tau_\Phi)$ is *finer* than $(\Psi, \tau_\Psi)$ and we write $(\Phi, \tau_\Phi) \geq (\Psi, \tau_\Psi)$, if $\Phi \subseteq \Psi$ and $\tau_\Phi \geq \tau_\Psi$. In particular, if $\tau_\Phi$ and $\tau_\Psi$ are finer than the topology induced by $\mathcal{H}$ in $\Phi$ and $\Psi$, respectively, we have:

$$\Phi \subseteq \Psi \subseteq \mathcal{H} \subseteq \Psi^\times \subseteq \Phi^\times.$$

This is a partial ordering in the class of riggings of $(\Lambda, \mathcal{A}, \mathcal{H}, P)$.

The next result shows the existence of minimal riggings.

**Theorem 15.** *Let $(\Lambda, \mathcal{A}, \mathcal{H}, P)$ be a spectral measure. Each direct integral of the form $\mathcal{H}_{\mu,N}$ associated to $(\Lambda, \mathcal{A}, \mathcal{H}, P)$, by the von Neumann theorem, along with one of its measurable orthonormal basis $\{e_k(\lambda)\}_{k=1}^{N(\lambda)}$, or equivalently, each generating system $\{g_k\}_{k=1}^m$ in $\mathcal{H}$ with respect to $P$ such that*

*a.* $[P] = [g_1]_P \succ [g_2]_P \succ \cdots,$
*b.* *if* $1 \leq j \leq k \leq m$, *then* $\mu_{g_j}|_{\Lambda(g_k)} = \mu_{g_k}$,

*provide a rigging $(\Phi, \tau, \mu, \lambda k^\lambda)$. This rigging is characterized by the following properties:*

*(i) The subspace $\Phi$ is dense in $\mathcal{H}$ and is given by*

$$\Phi = \left\{ \phi \in \mathcal{H} : \textit{ existe } \frac{d\mu_{\phi,g_k}}{d\mu_{g_k}}(\lambda) < \infty, \forall\lambda \in \Lambda\backslash\Lambda_0, \forall_k \in [1, N(\lambda)\rangle \right\},$$

*where $\Lambda_0$ is a subset of $\Lambda$ with $\mu$ zero measure (or equivalently, $P$ zero measure).*

*(ii) The complete family of antilinear functionals on $\Phi$, fulfilling (21), is of the form*

$$\{|\lambda k^\times\rangle : \lambda \in \Lambda\backslash\Lambda_0,\ k \in [1, N(\lambda)\rangle\},$$

*where we define each $|\lambda k^\times\rangle$ in terms of the isomorphism $V$ in Theorem 13:*

$$\langle\phi|\lambda k^\times\rangle = (V\phi(\lambda), e_k(\lambda))_{\mathcal{H}_\lambda} = \sqrt{\frac{d\mu_{g_k}}{d\mu}(\lambda)}\frac{d\mu_{\phi,g_k}}{d\mu_{g_k}}(\lambda), \quad \forall\phi \in \Phi. \quad (30)$$

*(iii) $\tau_\Phi$ is the weak topology $\sigma(\Phi, \Phi^\times)$, i.e., the coarsest compatible with the dual pair $(\Phi, \Phi^\times)$. The topological dual $\Phi^\times$ is the vector space spanned by the set $|\lambda k^\times$.*
*The topology $\tau_\Phi$ is produced by the following family of seminorms:*

$$\phi \mapsto |\langle\phi|\lambda k^\times\rangle|, \lambda \in \Lambda\backslash\Lambda_0,\ k \in [1, N(\lambda)\rangle.$$

*Then, the rigging* $(\Phi, \tau, \mu, \lambda^{\times})$ *is minimal. This means that no topology on* $\Phi$ *coarser than* $\tau$ *(except for the indeterminacy that produces the choice of the zero* $\mu$ *measure set* $\Lambda_0$*) can rig the spectral measure* $(\Lambda, \mathcal{A}, \mathcal{H}, P)$*.*

The proof of *Theorem 15* is based on the following idea: If we a priori know the antilinear functionals $|\lambda k^{\times}\rangle$ on $\Phi$, these vectors define the seminorms $\phi \mapsto |\langle\phi|\lambda k^{\times}\rangle|$. Then $|\lambda k^{\times}\rangle \in \Phi^{\times}$ if and only if the topology on $\Phi$ makes these seminorms continuous. In other words, the topology on $\Phi$ must be either equal to $\tau$ or stronger in order that $|\lambda k^{\times}\rangle \in \Phi^{\times}$.

### 6.3. Explicit Form of the Eigenoperators

The following result permits the identification of the Foias operators $I(\lambda)$ and $I^{\times}(\lambda)$:

**Theorem 16.** *Let* $(\Lambda, \mathcal{A}, \mathcal{H}, P)$, $\{g_k\}_{k=1}^m$, $\mathcal{H}_{\mu,N}$, $\{e_k(\lambda)\}_{k=1}^{N(\lambda)}$, $V$, $\Phi$ *and* $|\lambda k^{\times}\rangle$ *be as in Theorem 15. Then, the mappings:*

$$\begin{aligned} I(\lambda) : \Phi &\to \mathcal{H}_\lambda \\ \phi &\mapsto \sum_{k=1}^{m} \langle\phi|\lambda k^{\times}\rangle^* e_k(\lambda) \\ &= \sum_{k=1}^{m} (e_k(\lambda), V\phi(\lambda))_\lambda e_k(\lambda) \\ &= \sum_{k=1}^{m} \sqrt{\frac{d\mu_{g_k}}{d\mu}(\lambda)}\frac{d\mu_{g_k,\phi}}{d\mu_{g_k}}(\lambda) e_k(\lambda) \end{aligned} \tag{31}$$

*are well defined* $\mu$ *almost everywhere on* $\Lambda$*. The following relation holds:*

$$(P(E)I\phi, h) = \int_E (I(\lambda)\phi, Vh(\lambda))_\lambda d\mu(\lambda), \;\; \phi \in \Phi, h \in \mathcal{H}, E \in \mathcal{A}. \tag{32}$$

*In particular if* $E = \Lambda$,

$$(I\phi, h) = \int_\Lambda (I(\lambda)\phi, Vh(\lambda))_\lambda d\mu(\lambda), \;\; \phi \in \Phi, h \in \mathcal{H}. \tag{33}$$

*The adjoint operator* $I^{\times}(\lambda)$ *is*

$$\begin{aligned} I^{\times}(\lambda) : \mathcal{H}_\lambda &\to \Phi^{\times} \\ h_\lambda &\mapsto I^{\times}(\lambda)h_\lambda : \Phi \to \mathbb{C} \\ \varphi &\mapsto h_\lambda(I(\lambda)\varphi), \end{aligned}$$

*where*

$$h_\lambda(I(\lambda)\varphi) = \left(\sum_{k=1}^{m} \langle\varphi|\lambda k^\times\rangle^* e_k(\lambda), h_\lambda\right)_{\mathcal{H}_\lambda}$$

$$= \left(\sum_{k=1}^{m} e_k(\lambda), V\varphi(\lambda))_\lambda e_k(\lambda), h_\lambda\right)_{\mathcal{H}_\lambda}$$

$$= \left(\sum_{k=1}^{m} \sqrt{\frac{d\mu_{g_k}}{d\mu}(\lambda)} \frac{d\mu_{g_k,\varphi}}{d\mu_{g_k}}(\lambda) e_k(\lambda), h_\lambda\right)_{\mathcal{H}_\lambda}.$$

*In particular if* $h_\lambda = e_j(\lambda)$, *then* $I^\times(\lambda)e_j(\lambda) = |\lambda j^\times\rangle$

$$I^\times(\lambda) : \mathcal{H}_\lambda \to \Phi^\times$$

$$e_j(\lambda) \mapsto I^\times(\lambda)e_j(\lambda) : \Phi \to \mathbb{C}$$

$$\varphi \mapsto \langle\varphi|\lambda j^\times\rangle.$$

*The eigenoperators* $\gamma(\lambda) = I^\times(\lambda)I(\lambda)$ *of the spectral measure are of the form*

$$\gamma(\lambda) = I^\times(\lambda)I(\lambda) : \Phi \to \Phi^\times$$

$$\phi \mapsto \gamma(\lambda)\phi : \Phi \to \mathbb{C}$$

$$\varphi \mapsto [\gamma(\lambda)\phi](\varphi)$$

*where*

$$[\gamma(\lambda)\phi](\varphi) = \left(\sum_{k=1}^{m} \langle\phi|\lambda k^\times\rangle e_k(\lambda), \sum_{k=1}^{m} \langle\varphi|\lambda k^\times\rangle e_k(\lambda)\right)_{\mathcal{H}_\lambda}$$

$$= \sum_{k=1}^{m} \langle\phi|\lambda k^\times\rangle^* \langle\varphi|\lambda k^\times\rangle.$$

*The eigenforms of the spectral measure* $\phi^\times = \gamma(\lambda)\phi \in \Phi^\times$, *where* $\phi \in \Phi$ *are given by:*

$$\phi^\times : \Phi \to \mathbb{C}$$

$$\varphi \mapsto \sum_{k=1}^{m} \langle\phi|\lambda k^\times\rangle^* \langle\varphi|\lambda k^\times\rangle,$$

*In particular, if we choose* $\phi$ *such that* $I\phi \in \mathcal{H}_{g_j}$, *then,*

$$\phi^\times(\varphi) = \langle\phi|\lambda j^\times\rangle^* \langle\varphi|\lambda j^\times\rangle, \ \forall_\varphi \in \Phi.$$

Once we have identified the eigenoperators and the eigenforms, the topologies on $\Phi$ for which there exist an integral decomposition of the spectral measure can be explicity determined:

**Theorem 17.** *With the hypothesis of Theorem 16, we have that:*

*i. A sufficient condition for the mappings $I(\lambda)$ in (31) be continuous with respect to a topology $\tau$ defined on $\Phi$ is that for $\lambda \in \Lambda$ (save for a set with zero $\mu$ measure) the family of antilinear forms $\{|\lambda k^\times\rangle : k = 1, 2, \ldots\}$ be equicontinuous with respect to $\tau$.*

*ii. If $m = \mu$-sup $N(\lambda) = P$-sup $N_P(\lambda)$ is finite, then the mappings $I(\lambda)$ are continuous for the minimal topology $\tau_\Phi$ as defined in Theorem 13.*

The continuity of the eigenoperators and eigenforms can be derived from the continuity of the mapping $I(\lambda)$ and the following proposition:

**Proposition 18.** *Under the conditions of Theorem 16, if we endow $\Phi$ with a locally convex topology for which $I(\lambda)$ is continuous and let $\phi^\times_\beta$ the antidual space with the strong topology $\beta(\Phi^\times, \Phi)$. Then,*

*i. The mapping $I^\times(\lambda) : \mathcal{H}_\lambda \to \Phi^\times$ is weakly and strongly continuous, i.e., it is $(\sigma(\mathcal{H}, \mathcal{H}), \sigma(\Phi^\times, \Phi))$ and $\|\cdot\|_{\mathcal{H}_\lambda}, \beta(\Phi^\times, \Phi)$ continuous.*

*ii. The eigenoperator $\gamma(\lambda) = I^\times(\lambda) I(\lambda) : \Phi \to \Phi^\times$ belongs to $\mathcal{L}(\Phi, \Phi^\times_\beta)$.*

### 6.4. Kato-Kuroda Riggings

In the next result we establish the connections between the spectral representations of Kato–Kuroda (*theorem 9*) and the concretion of Birman–Solomjak (*proposition 11*).

**Theorem 19.** *Let $(\Lambda, \mathcal{A}, \mathcal{H}, P)$ be a spectral measure and let $\{g_k\}_{k=1}^m$ be a generating system, $\mathcal{H}_{\mu,N}$ a direct integral associated to $P$, $\{e_j(\lambda)\}_{k=1}^m$ an orthonormal measurable basis and the operator $V : \mathcal{H} \to \mathcal{H}_{\mu,N}$ as in section 3.2. (We suppose that $\mu$ is a $\sigma$-finite nonnegative scalar measure on $(\Lambda, \mathcal{A})$ of the same type as $[P]$.)*

*On the other hand, let us consider the spectral system $(\Lambda, \mathcal{A}, \mu, \mathcal{H}, P)$, a spectral form $(\hat{\Lambda}, \Phi, s)$ for it, the direct integral $\mathcal{H}_{\mu,\Phi}$, and an operator $V' : \mathcal{H} \to \mathcal{H}_{\mu,\Phi}$ as in theorem 9.*

Then, if $\{g_k\}_{k=1}^m \subset \Phi$, for each $h \in \mathcal{H}$ and $\lambda \in \hat{\Lambda}$ we have

$$[Vh](\lambda) = \oplus_{j=1}^m \sqrt{\frac{d\mu_{g_1}}{d\mu}(\lambda)} \frac{d\mu_{g_j,h}}{d\mu_{g_j}}(\lambda)\, e_k(\lambda) \tag{34}$$

and

$$[V'h](\lambda) = \oplus_{j=1}^m \frac{d\mu_{g_j,h}}{d\mu_{g_j}}(\lambda)\, q_\lambda g_j. \tag{35}$$

We note two consequences of *theorem 19*:

(i) In the concretion of Kato–Kuroda (formula (35)) the normalization factors $\sqrt{\frac{d\mu_{g_1}}{d\mu}(\lambda)}$ don't appear, or better, they are included in $q_\lambda g_j$. This is the relevant advantage of that construction.

(ii) Under the conditions of the theorem, The isomorphisms $\mathcal{F}V^{-1}$ and $V\mathcal{F}V^{-1}$ between the direct integrals $\mathcal{H}_{\mu,\Phi}$ y $\mathcal{H}_{\mu,N}$ relate the measurable families

$$\{g_\lambda g_j\}_{\lambda\in\hat{\Lambda}} \ y \ \left\{\sqrt{\frac{d\mu_{g_1}}{d\mu}(\lambda)}\, e_j(\lambda)\right\}_{\lambda\in\hat{\Lambda}},$$

where $j \in [1, m\rangle$.

Now, we consider the eigenfunction expansions of Kato–Kuroda. We are going to see that under the Birman–Solomjak conditions in section 3.2 it is possible to obtain explicitly a complete system of eigenfunctions at the same time than the eigenvectors of Marlow (1965) and the eigenoperators of Foias (1959a,b, 1962).

An orthonormal measurable basis $\{e_j\}$ of the direct integral

$$\mathcal{H}_{\mu,N} = \int_\Lambda^\oplus \mathcal{H}_\lambda \, d\mu$$

induces a unitary isomorphism between the spaces

$$\mathcal{H}_{\mu,N} \simeq L^2(\Lambda_\infty, \mu; l^2) \oplus \left[\, \oplus_1^\infty L^2(\Lambda_m, \mu; \mathbb{C}^m)\right],$$

where the sets $\Lambda_k$ are

$$\Lambda_k = \{\lambda \in \Lambda : N(\lambda) = k\}, \quad k \in [1, \infty\rangle.$$

On the other hand, if we consider the discrete measure $d$ on $\mathbb{N}$ and $[1, m\rangle$, where $m \in \mathbb{N}$, we have the following unitary isomorphisms:

$$L^2(\Lambda_\infty, \mu; l^2) \simeq L^2(\Lambda_\infty \times \mathbb{N}, \mu \times d)$$

and

$$L^2(\Lambda_m, \mu; \mathbb{C}^m) \simeq L^2(\Lambda_m, \times[1, m\rangle, \mu \times d).$$

Finally, in the union

$$(\Lambda_\infty \times \mathbb{N}) \bigcup \left[\bigcup_{m=1}^\infty (\Lambda_m \times [1, m\rangle)\right]$$

we define the measure $\nu$ as the sum of the measures $\mu \times d$ that we consider above in each of them.

**Theorem 20.** *Let the spectral system* $(\Lambda, \mathcal{A}, \mu, \mathcal{H}, P)$, *the spectral form* $(\hat{\Lambda}, \Phi, s)$ *for it, the generating system* $\{g_k\}_{k=1}^m$ *for* $P$, *the direct integral* $\mathcal{H}_{\mu,\mathbb{N}}$ *the orthonormal measurable basis* $\{e_j(\lambda)\}_{k=1}^m$, *and the operator* $V : \mathcal{H} \to \mathcal{H}_{\mu,\mathbb{N}}$ *be as in section 3.2. Let* $\Phi$ *be a minimal rigging (see section 6.2), where* $\hat{\Lambda} = \Lambda \backslash \Lambda_0$ *and* $\{g_k\}_{k=1}^m \subset \Phi$. *Then in the space*

$$L^2\left((\hat{\Lambda}_\infty \times \mathbb{N}) \bigcup \left[\bigcup_{m=1}^{\infty} (\hat{\Lambda}_m \times [1, m\rangle)\right], \nu\right),$$

*where* $\hat{\Lambda}_s = \Lambda_s \bigcap \hat{\Lambda}$ *for all* $s \in \{\infty\} \cup \mathbb{N}$, *we have a complete system of eigenfunctions of the form*

$$\begin{aligned}[\psi^\times(\lambda \times k)]\phi &= (e_k(\lambda), V_\phi(\lambda))_\lambda \\ &= \sqrt{\frac{d\mu_{g_k}}{d\mu}(\lambda)} \frac{d\mu_{g_k,\phi}}{d\mu_{g_k}}(\lambda). \end{aligned} \tag{36}$$

*(These identities are verified for every* $\phi \in \Phi$ *and every* $(\lambda, k) \in (\hat{\Lambda}_\infty \times \mathbb{N}) \bigcup [\bigcup_{m=1}^{\infty} (\hat{\Lambda}_m \times [1, m\rangle)]$*).*

*On the other hand, when* $\Phi = span\, \{g_j : j \in [1, m\rangle\}$, *only the normalization factors are relevant: In this case, each* $\phi \in \Phi$ *is of the form (finite sum)* $\phi = \sum c_j g_j$, *where* $c_j \in \mathbb{C}$, *and then*

$$[\psi^\times(\lambda \times k)]\phi = c_k \sqrt{\frac{d\mu_{g_k}}{d\mu}(\lambda)}, \quad \lambda \in \hat{\Lambda}, k \in [1, m\rangle. \tag{37}$$

We can use the approximate spectral forms as a tool to construct locally convex riggings of a spectral measure. Under the conditions of *Theorem 20*, we know that the generalized eigenvectors of the spectral measure $P$ are of the form

$$\langle \phi | \lambda k^\times \rangle = \sqrt{\frac{d\mu_{g_k}}{d\mu}(\lambda)} \frac{d\mu_{\phi,g_k}}{d\mu_{g_k}}(\lambda),$$

for all $\phi \in \Phi$, $\lambda \in \hat{\Lambda}$ and $k \in [1, N(\Lambda)\rangle$. In this case, we can write the spectral form in the following terms

$$s(\lambda; \phi, \varphi) = \sum_{k=1}^{m} \langle \phi | \lambda k^\times \rangle \langle \lambda k^\times | \varphi \rangle,$$

for each $\phi, \varphi \in \Phi$ and $\lambda \in \hat{\Lambda}$. In particular, if $\varphi = g_k$, then

$$s(\lambda; \phi, g_k) = \sqrt{\frac{d\mu_{g_k}}{d\mu}(\lambda)} \langle \phi | \lambda k^\times \rangle.$$

Now, if $s_\epsilon$ is an approximate spectral form for $(\hat{\Lambda}, \Phi, s)$, we have

$$\lim_{\epsilon \to 0} s_\epsilon(\lambda; \phi, g_k) = \sqrt{\frac{d\mu_{g_k}}{d\mu}}(\lambda)\langle \phi | \lambda k^\times \rangle. \tag{38}$$

Finally, we consider on $\Phi$ a topology of locally convex topological vector spaces for which the approximate spectral form $s_\epsilon$ is equicontinuous with respect to $\epsilon$ for each $\lambda \in \hat{\Lambda}$. Then the generalized eigenvectors $\lambda k^\times$ belong to $\Phi^\times$.

*Definition 21.* We say that a locally convex rigging $(\Phi, \tau_\Phi)$ of a spectral measure space $(\Lambda, \mathcal{A}, \mathcal{H}, P)$ is a *Kato–Kuroda rigging* if there is an approximate spectral form $s_\epsilon$ equicontinuous with respect to $\epsilon$ for each $\lambda \in \Lambda \backslash \Lambda_0$. (Here $\Lambda_0$ is the set considered in the definition of the rigging).

## 6.5. Inductive and Nuclear Versions of the Spectral Theorem

To finish our presentation, we give two new versions of both the nuclear and the inductive spectral theorems. In these versions, this kind of spaces, nuclear and inductive limit, appear as universal riggings of Vitali spectral measures, i.e., these spaces equip any Vitali spectral measure. This happens in particular for the absolutely continuous part, with respect to the Lebesgue measure, and the discrete part of every normal operator.

### 6.5.1. *Vitali spectral measures*

The spectral measure given by $(\Lambda, \mathcal{A}, \mathcal{H}, P)$ is a *Vitali spectral measure* if being given its continuous part $(\Lambda, \mathcal{A}, \mathcal{H}_c, P_c)$, there exists a measure $\mu$ on $\Lambda$ with $[\mu] = [P]$, such that the measure space $(\Lambda, \mathcal{A}, \mu)$ admits a Vitali system.

If $(\Lambda, \mathcal{A}, \mathcal{H}, P)$ is a Vitali spectral measure, the Vitali Lebesgue theorem guarantees that if $g, h \in \mathcal{H}_c$, for almost all $\lambda \in \Lambda$ the Radon–Nikodym derivative $\frac{d\mu_{h,g}}{d\mu_g}(\lambda)$ exists and is equal to $\lim_{n\to 0} \frac{\mu_{h,g}(E_n)}{\mu_g(E_n)}$ for any sequence of set $E_n$ admitting a contraction to $\lambda$.

*Example.* Every spectral measure defined on $(\mathbb{R}^n, \mathcal{B})$ without continuous singular part with respect to the Lebesgue measure is a Vitali spectral measure.

### 6.5.2. *The inductive version of the spectral theorem*

The inductive limit $(\mathcal{H}_n, I_n)^{(n\in\mathbb{N})}$ of a countable system where $\mathcal{H}_n$ are separable Hilbert spaces (where $\mathcal{H}_n \subset \mathcal{H}$) such that the identity mappings $I_n : \mathcal{H} \mapsto \mathcal{H}$ are Hilbert–Schmidt is a universal rigging.

**Theorem 22.** *Let $\mathcal{H}$ be a separable Hilbert space and*

$$(\mathcal{H}_n, I_n)^{(n\in\mathbb{N})}$$

*an inductive system for which each $\mathcal{H}_n$ is a separable Hilbert space and the identity mappings $I_n : \mathcal{H} \mapsto \mathcal{H}$ are Hilbert-Schmidt for all $n \in \mathbb{N}$. If*

$$\Phi = \mathrm{span}\{\bigcup_{(n\in\mathbb{N})} \mathcal{R}(I_n)\}$$

*is dense in $\mathcal{H}$ and $\tau_l$ is the inductive topology produced by the system $(\mathcal{H}_n, I_n)^{(n\in\mathbb{N})}$ on $\Phi$, then, $(\Phi, \tau_l)$ rigs any Vitali spectral measure $(\Lambda, \mathcal{A}, \mathcal{H}, P)$. In particular, $(\Phi, \tau_l)$ rigs the absolutely continuous and discrete parts of any spectral measure of the form $(\mathbb{C}, \mathcal{B}, \mathcal{H}, P)$.*

### *6.5.3. Nuclear version of the spectral theorem*

The original version of the Gelfand–Maurin nuclear spectral theorem (Gelfand and Vilenkin, 1964; Maurin, 1968) assumed that $\Phi$ is endowed with a nuclear topology, the canonical injection $I : \Phi \to \mathcal{H}$ is continuous, and therefore nuclear, and $A$ reduces $\Phi(A\Phi \subset \Phi)$ and is continuous on $\Phi$.

The present version of the nuclear spectral theorem uses the relation between the spectral measures and the direct integral of Hilbert spaces.

To begin with, let us write the following Lemma, due to Roberts (1966a,b):

**Lemma.** Let $\Phi$ be a locally convex topological vector space and let $T$ be a nuclear operator $T : \Phi \mapsto \mathcal{H}$, where $\mathcal{H}$ is a Hilbert space. Then, there exists a separable Banach space $X$ and two operators $T_1 : \Phi \mapsto X$ and $T_2 : X \mapsto \mathcal{H}$ such that $T_1$ is continuous, $T_2$ is nuclear, and $T = T_2 \circ T_1$.

Our version of the nuclear spectral theorem is the following:

**Theorem 23.** *Let $\mathcal{H}$ be a Hilbert space, $\Phi$ a dense subspace in $\mathcal{H}$, and $\tau_\Phi$ a nuclear topology on $\Phi$ such that the canonical injection $I : \Phi \mapsto \mathcal{H}$ is continuous. Then, $(\Phi, \tau_\Phi)$ rigs any Vitali spectral measure $(\Lambda, \mathcal{A}, \mathcal{H}, P)$. In particular, $(\Phi, \tau_\Phi)$ rigs the absolutely continuous and discrete parts of any spectral measure of the form $(\mathbb{C}, \mathcal{B}, \mathcal{H}, P)$.*

## 7. CONCLUDING REMARKS

As concluding remarks, we present two tables that summarize the present work. They are labeled as Table I and Table II. In Table I, we summarize the state of the art of the subject before the present research. On the other hand, Table II shows how the new framework of locally convex equipments of spectral measures as here introduced unifies the formalisms in Table I.

**Table I.** Dirac Kets

*On Hilbert spaces*

"In the conventional von Neumann–Mackey formulation,
a quantum system is described in the language of Hilbert space . . ."

| | | |
|---|---|---|
| Projection-valued measures $P$ on Hilbert spaces $\mathcal{H}$.<br>Spectral measure spaces $(\Lambda, \mathcal{A}, \mathcal{H}, P)$. | → | Only with discrete spectrum.<br>*Eigenvectors* in $\mathcal{H}$.<br>Classical spectral theorem.<br>Wave packets. |
| Direct integral decompositions of Hilbert spaces $\mathcal{H}_{\mu,N} = \int_\Lambda^\oplus \mathcal{H}_\lambda d\mu(\lambda)$. | → | Continuous and discrete spectrum.<br>*Eigenvectors* in $\mathcal{H}_\lambda$.<br>Functional spectral theorem.<br>Only if $\lambda \in \sigma_p$, $\mathcal{H}_\lambda \subseteq \mathcal{H}$ and $P_\lambda : \mathcal{H} \to \mathcal{H}_\lambda$ is continuous. |

*On auxiliary topological vector spaces*

. . . One would like to go beyond Hilbert space in order to be able to incorporate very singular objects. But at the same time, one wants to keep the good geometrical structure of Hilbert space, and the spectral theory as well, that fits so neatly with the interpretation of quantum mechanics. The answer is to consider a structure built around a Hilbert space, in the spirit of distribution theory." Antoine [1998].

| | | |
|---|---|---|
| Gelfand triplets.<br>Rigged Hilbert spaces.<br>$\Phi \subset \mathcal{H} \subset \Phi^\times$.<br>Integral decompositions of Foias. | → | Mainly when $\Phi$ is a nuclear tvs.<br>*Eigenoperators* in $\mathcal{L}(\Phi, \Phi_\beta^\times)$.<br>*Eigenforms* in $\Phi^\times$.<br>Nuclear spectral theorem. |
| Eigenfunction expansions of Kato–Kuroda on Auxiliary pairs $(\Phi, \Phi^\times)$. | → | *Eigenfunctions* in $\Phi^\times$.<br>Spectral representations on Direct integrals from $\Phi$. |
| Other Formalisms → (See Antoine (1998).) | | Scales of Hilbert or Banach spaces, Lattices of Hilbert or Banach spaces, Partial inner product spaces. |

**Table II.** Locally Convex Equipments

| A Unified Formalism | | |
|---|---|---|
| Spectral Measure Spaces<br>$(\Lambda, \mathcal{A}, \mathcal{H}, P)$.<br>Direct integral decompositions of<br>Hilbert spaces<br>$\mathcal{H}_{\mu,N} = \int_\Lambda^\oplus \mathcal{H}_\lambda \, d\mu(\lambda)$. | $\leftrightarrow$ | *Minimal equipments*<br>• Tight riggings: They adjust their structure to a concrete spectral measure or direct integral.<br>• The topology of $\Phi$ is $\sigma(\Phi, \Phi^\times)$ and $\Phi^\times$ is generated by the eigenvectors.<br>• Identification of eigenvectors. |
| Gelfand triplets.<br>Rigged Hilbert spaces.<br>$\Phi \subset \mathcal{H} \subset \Phi^\times$.<br>Integral Decompositions of Foias. | $\leftrightarrow$ | *Universal equipments*<br>• These riggings equip every Vitali spectral measure.<br>• New versions of the nuclear and inductive spectral theorem.<br>• Identification of the eigenoperators and the eigenforms. |
| Eigenfuction expansions of<br>Kato–Kuroda on Auxiliary<br>pairs $(\Phi, \Phi^\times)$. | $\leftrightarrow$ | *Equipments of Kato–Kuroda*<br>• Stationary methods and approximate spectral forms.<br>• Identification of eigenfunctions. |

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